A General Signal of a Phase Transition from Single-Particle Momentum Distributions

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A two-particle space correlation function is derived from the single-particle momentum distribution of the emission source. A signal of a first order phase transition is obtained from this correlation function if density fluctuations are large.

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The search for phase transitions is a hot topic in atomic and nuclear physics. Several different phase transitions are observed/or expected to be observed like ^{3}He to ^{4}He superfluid, Bose-Einstein condensation, liquid to gas nuclei and hadronic matter to Quark Gluon Plasma (QGP). The QGP phase transition is predicted by lattice QCD and several QGP signals have been suggested[1], although no convincing evidence[2] has yet been presented. The nature of the phase transition of QGP depends on the dynamical quarks. A first-order phase transition is expected without dynamical quarks. Simulations of lattice QCD with dynamical quarks have not yet overcome the limitations due to finite lattice size. Present results indicate a smooth cross-over between phases for two light quark flavours (N_f=2) and a first-order phase transition for N_f >3[1].

Among these QGP signals there are two based on the study of particle average transverse momentum ($\langle p_T \rangle$) and on two particle momentum correlations. If the dependence of $\langle p_T \rangle$ on the (pseudo-) rapidity density ($dn/d\eta$) of produced particles[3] is studied, the phase transition of QGP to hadronic matter could be seen as a plateau followed by a second rise in $\langle p_T \rangle$. However this signal is not sensitive to the shape of the single-particle momentum distribution. When studying the emission source space-time extension through Bose-Einstein correlations, it was argued that the two particle momentum correlations can signal a QGP phase transition[4] through a very large apparent source radius in the 'outward' direction.

Recently much attention has been paid to search for disoriented chiral condensates (dcc) from single-particle momentum distributions and two-particle momentum correlations[5]. Bjorken et al. pointed out that large fluctuations in the pion spectra can be expected from the dcc [6] which create large variation in the source density.

The fluctuation of the source density near critical point was first discussed by Ornstein and Zernike[7]. In this paper we study the two-particle space correlations of the emission source through density fluctuations in heavy ion physics. The two-particle space correlation function is derived from the single-particle momentum distributions. A signal of a first order phase transitions sensitive to the shape of the single particle momentum distribution is proposed. This signal is not only suitable for searching of the QGP phase transition but also for the search of the liquid to gas phase transition in the excited nucleus.

Assume a source at rest with volume V in which there are N particles distributed in a set of positions $\vec{r}_i$, $i = 1, 2, ...N$. The normalized N particle probability distribution is $p(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$. The particle number density at a position $\vec{r}$ is defined as,

$$\rho_1(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i). \tag{1}$$

Here δ represents a Dirac delta function. Thus $\rho_1(\vec{r})d\vec{r}$ is the number of particles between $\vec{r}$ - $\vec{r} + d\vec{r}$. The integration of the source volume will give

$$\int_V \rho_1(\vec{r})d\vec{r} = N. \tag{2}$$

The ensemble average of the particle number density $\overline{\rho_1(\vec{r})}$ is assumed to be independent on the position $\vec{r}$ and denoted as ρ,

$$\begin{aligned}\rho &= \int d\vec{r}_1 ... \int d\vec{r}_N \sum_{i=1}^{N} \delta(\vec{r}-\vec{r}_i) p(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) \\ &= N \int d\vec{r}_1 \delta(\vec{r}-\vec{r}_1) P(\vec{r}_1) \\ &= N\mathbf{P}(\vec{\mathbf{r}}). \end{aligned} \tag{3}$$

The $\rho_1(\vec{r})$ is the local particle density which fluctuates around the ρ and $\rho_1(\vec{r})-\rho$ reflects the density fluctuation of the source. The volume integral of ρ gives the average number of particles $\langle N\rangle$

$$\int d\vec{r} \rho(\vec{r}) = \langle N\rangle. \tag{4}$$

The two-particle density correlation function can be expressed by,

$$\rho_2(\vec{r}, \vec{r'}) = \sum_{i\neq j=1} \delta(\vec{r}-\vec{r}_i)\delta(\vec{r'}-\vec{r}_j). \tag{5}$$

The ensemble average of $\rho_2(\vec{r}, \vec{r'})$ is obtained in a similar way as $\rho(\vec{r})$,

$$\overline{\rho_2(\vec{r}, \vec{r'})} = N(N-1)P(\vec{r}, \vec{r'}). \tag{6}$$

The integral of $\overline{\rho_2(\vec{r}, \vec{r'})}$ is the mean number of two particle pairs $\langle N(N-1)\rangle$ in the volume.

The density-density correlation is defined as $\overline{\rho_1(\vec{r})\rho_1(\vec{r'})}$. From the formulas above it is found

$$\overline{\rho_1(\vec{r})\rho_1(\vec{r'})} = \rho\delta(\vec{r}-\vec{r'}) + \overline{\rho_2(\vec{r}, \vec{r'})}. \tag{7}$$

One can now study the density-density fluctuations from the density-density correlation,

$$\overline{[\rho_1(\vec{r})-\rho][\rho_1(\vec{r'})-\rho]} = \rho\delta(\vec{r}-\vec{r'}) + \overline{\rho_2(\vec{r}, \vec{r'})} - \rho^2. \tag{8}$$

The Fourier transformation of $\rho_1(\vec{r})-\rho$, is,

$$\rho_1(\vec{r})-\rho = (2\pi)^{-3} \int \rho_1(\vec{k}) \exp(i\vec{k}\cdot\vec{r}) d\vec{k}. \tag{9}$$

Where $\rho_1(\vec{k})$ is the Fourier component of particle number density fluctuation which is a complex variable. Because $\rho_1(\vec{r})-\rho$ is real, $\rho_1^*(\vec{k}) = \rho_1(-\vec{k})$ and because of the normalization $\rho_1(\vec{r})$, $\rho_1(\vec{k})|_{\vec{k}=0} = 0$. Equations 8 and 9 give

$$\overline{|\rho_1(\vec{k})|^2} = V\rho + V \int e^{-i\vec{k}\cdot\vec{r}} (\overline{\rho_2(\vec{r})} - \rho^2) d\vec{r}, \tag{10}$$

by assuming that the two particle space correlation only depends on the distance between two particles. The statistical average on the left side could be found in the following way. The total free energy F is expressed in terms of the free-energy density $F_1(\vec{r})$,

$$F = \int F_1(\vec{r}) d\vec{r}. \tag{11}$$

For a given temperature, $F_1(\vec{r})$ may deviate from its equilibrium value $\overline{F_1}$ due to the density fluctuations. Because $F_1(\vec{r})$ should be minimum for equilibrium, Let us expand the difference $F_1(\vec{r}) - \overline{F_1}$ according to Landau's approach[8]:

$$F_1(\vec{r}) - \overline{F_1} = \frac{1}{2} a[\rho_1(\vec{r})-\rho]^2 + \frac{1}{2} b(\nabla\rho_1(\vec{r}))^2 + ... \tag{12}$$

here parameters a and b are independent of the density but may depend on the temperature. For a finite size source the Fourier integral should be replaced by the Fourier series. Neglecting higher-order terms in eq.(12),

$$F - \overline{F_1}V = \frac{1}{2V} \sum (a + b\vec{k}^2)|\rho_1(\vec{k})|^2. \tag{13}$$

Based on this expression, the probability of having the fluctuation $|\rho_1(\vec{k})|$ is given by,

$$p(|\rho_1(\vec{k})|) = A\exp(-(a+b\vec{k}^2)|\rho_1(\vec{k})|^2/2VT). \tag{14}$$

Here $A = (\int e^{-(a+b\vec{k}^2)|\rho_1(\vec{k})|^2/2VT} d\rho_1(\vec{k}))^{-1}$ which implies,

$$\overline{|\rho_1(\vec{k})|^2} = \frac{VT}{a+b\vec{k}^2}. \tag{15}$$

Using this and taking the Fourier inverse in eq.(10), we obtain two particle space correlation function

$$\overline{\rho_2(r)} - \rho^2 + \rho\delta(r) = \frac{T}{4\pi br} \exp[-(\frac{a}{b})^{1/2} r] \tag{16}$$

The parameter a must vanish at the phase transition point (critical point $T = T_c$) with zero net-baryon density, because

$$a = (\frac{\partial^2 F_1(r)}{\partial\rho_1(r)^2})_T = \frac{1}{\rho_1(r)} (\frac{\partial P}{\partial\rho_1(r)})_T. \tag{17}$$

Here P is the pressure of the system. For the first-order phase transition, the two-particle space correlation function becomes long-range,

$$\overline{\rho_2(r)} - \rho^2 + \rho\delta(\vec{r}) = \frac{T_c}{4\pi br}, \quad T = T_c. \tag{18}$$

In the vicinity of T_c, a is non-zero. Thus the two-particle space correlation function decreases exponentially.

Eq.(16) shows that two-particle space correlation function contains information of the first order phase transition. Now the way of obtaining such information experimentally will be presented in accordance with the second quantisation method.

Assume that particles with $\vec{p} = \hbar\vec{k}$ move freely in a static source of volume V at freeze-out. The normalized single-particle wave function is

$$\phi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}} \exp(i\vec{k}\cdot\vec{r}). \tag{19}$$

The operators $\hat{a}^+_{\vec{k}}$ and $\hat{a}_{\vec{k}}$ increase and decrease in units of the numbers $n_{\vec{k}}$ of particles in the various quantum states $\phi_{\vec{k}}$. The operators become,

$$\hat{\Psi}^+(\vec{r}) = \sum \phi^*_{\vec{k}}(\vec{r})\hat{a}^+_{\vec{k}}, \quad \hat{\Psi}(\vec{r}) = \sum \phi_{\vec{k}}(\vec{r})\hat{a}_{\vec{k}} \tag{20}$$

which respectively add and remove one particle from point $\vec{r}$ in the system. The operator $\hat{\Psi^+}(\vec{r})\hat{\Psi}(\vec{r})$dV is the operator of the number of particles in the volume dV. Hence $\hat{\Psi}^+\hat{\Psi}$ can be regarded as an operator $\hat{n}$, which represents the particle density distribution in space

$$\hat{n} = \hat{\Psi}^+(\vec{r})\hat{\Psi}(\vec{r}) = \sum_{\vec{k}}\sum_{\vec{k}'} \hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}'}\phi^*_{\vec{k}}\phi_{\vec{k}'}. \tag{21}$$

The diagonal terms of the sum ($\vec{k} = \vec{k}'$) give the mean density $\overline{n}$ in the quantum state considered,

$$\begin{aligned} \overline{n} &= <\vec{k}|\sum_{\vec{k}} \hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}|\phi_{\vec{k}}|^2|\vec{k}> \\ &= \frac{1}{V}\sum <\vec{k}|\hat{n}_{\vec{k}}|\vec{k}> = \frac{1}{V}\sum n_{\vec{k}}. \end{aligned} \tag{22}$$

Here operator $\hat{n}_{\vec{k}} = \hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}$ and $n_{\vec{k}}$ is the number of particles in the quantum state. The operator which contributes to the density fluctuation is

$$\hat{n} - \overline{n} = \sum_{\vec{k}}\sum_{\vec{k}'\neq\vec{k}} \hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}'}\phi^*_{\vec{k}}\phi_{\vec{k}'}. \tag{23}$$

The average density-density fluctuation is marked as $\overline{(\hat{n}(\vec{r}_1)-\overline{n})(\hat{n}(\vec{r}_2)-\overline{n})}$. This mean value is calculated in two stages. First of all, the quantum averaging over all the quantum states of $\phi_{\vec{k}}$ and $\phi_{\vec{k}'}$ then the ensember (statistics) averaging should be carried out.

The average quantum state is as

$$\begin{aligned} & D2 \\ = \ & \langle \vec{k}_1\vec{k'}_1\vec{k}_2\vec{k'}_2|(\hat{n}(\vec{r}_1)-\overline{n})(\hat{n}(\vec{r}_2)-\overline{n})|\vec{k}_1\vec{k'}_1\vec{k}_2\vec{k'}_2\rangle \\ = \ & \langle \vec{k}_1\vec{k'}_1\vec{k}_2\vec{k'}_2|\sum_{\vec{k}_1}{\sum_{\vec{k'}_1}}'\sum_{\vec{k}_2}{\sum_{\vec{k}'_2}}' \hat{a}^+_{\vec{k}_1}\hat{a}_{\vec{k'}_1}\hat{a}^+_{\vec{k}_2}\hat{a}_{\vec{k'}_2} \\ & \phi^*_{\vec{k}_1}(\vec{r}_1)\phi_{\vec{k'}_1}(\vec{r}_1)\phi^*_{\vec{k}_2}(\vec{r}_2)\phi_{\vec{k'}_2}(\vec{r}_2)|\vec{k}_1\vec{k'}_1\vec{k}_2\vec{k'}_2\rangle. \end{aligned} \tag{24}$$

$D2$ is not zero although $\vec{k}_1 = \vec{k'}_2$ and $\vec{k'}_1 = \vec{k}_2$. Let $\vec{k} = \vec{k}_1$ and $\vec{k'} = \vec{k}_2$, then the average quantum state of eq.(24) will become,

$$D2 = \frac{1}{V^2}\sum_{\vec{k}}\sum_{\vec{k'}}(1 \pm n_{\vec{k'}})n_{\vec{k}}\ e^{i(\vec{k}-\vec{k'})\cdot(\vec{r}_1-\vec{r}_2)}. \tag{25}$$

Here $n_{\vec{k}}$ is the number of particles in the quantum state. The '+' refers to the case of Bose statistics and '-' to that of Fermi statistics. After the quantum averaging the $D2$ must also be averaged in the statistical sense, i.e. over the equilibrium distribution of the particles in the various quantum states. Since particles in different quantum states are quite independent, the numbers $n_{\vec{k}}$ and $n_{\vec{k'}}$ are averaged independently,

$$\overline{(1 \pm n_{\vec{k'}})n_{\vec{k}}} \approx (1 \pm \overline{n}_{\vec{k'}})\overline{n}_{\vec{k}}. \tag{26}$$

If one now replace the summation in eq.(25) into integration, we obtain the following expression for the mean density-density fluctuation,

$$\begin{aligned} & \overline{(\hat{n}(\vec{r}_1)-\overline{n})(\hat{n}(\vec{r}_2)-\overline{n})} \\ = \ & \frac{1}{(2\pi)^3}\int \overline{n}_{\vec{k}}\ e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\delta(\vec{r}_1-\vec{r}_2)d\vec{k} \\ \pm \ & \frac{1}{(2\pi)^6}\int\int \overline{n}_{\vec{k'}}\overline{n}_{\vec{k}}\ e^{i(\vec{k}-\vec{k'})\cdot(\vec{r}_2-\vec{r}_1)}d\vec{k}d\vec{k'} \\ = \ & \overline{n}\delta(\vec{r}) \pm |\frac{1}{(2\pi)^3}\int \overline{n}_{\vec{k}}\ e^{i\vec{k}\cdot\vec{r}}d\vec{k}|^2, \end{aligned} \tag{27}$$

here $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\overline{n}_{\vec{k}} = \frac{1}{e^{(\varepsilon-\mu)/T}\mp 1}$ is the Bose/Fermi distribution.

After comparison of eq.(27) with eq.(8) the two-particle space correlation function is found to be,

$$\overline{\rho_2(\vec{r}_1,\vec{r}_2)} - \rho^2 = \pm|\frac{1}{(2\pi)^3}\int \overline{n}_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}d\vec{k}|^2. \tag{28}$$

For a classical (Boltzmann) system, the $\hat{a}_{\vec{k}}\hat{a}^+_{\vec{k}} = \hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}$, so that,

$$\begin{aligned} & \overline{(\hat{n}(\vec{r}_1)-\overline{n})(\hat{n}(\vec{r}_2)-\overline{n})} \\ = \ & \frac{1}{(2\pi)^6}|\int \overline{n}_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}d\vec{k}|^2, \end{aligned} \tag{29}$$

and the two particle space correlation function is

$$\overline{\rho_2(\vec{r}_1,\vec{r}_2)} - \rho^2 + \rho\delta(\vec{r}_1 - \vec{r}_2) = \frac{1}{(2\pi)^6}|\int \overline{n}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} d\vec{k}|^2. \tag{30}$$

Here $\overline{n}_{\vec{k}} = \exp((\mu - \varepsilon)/T)$ is the Boltzmann distribution.

If the spin S of the particles is taken into account, the two particle space correlation function will become

$$\overline{\rho_2(\vec{r}_1,\vec{r}_2)} - \rho^2 = \pm\frac{g^2}{(2\pi)^6}|\int \overline{n}_{\vec{k},\sigma} e^{i\vec{k}\cdot\vec{r}} d\vec{k}|^2. \tag{31}$$

Here σ is the spin projection and $g = 2S + 1$ is the degenerace of the particle. $\overline{n}_{\vec{k},\sigma}$ is the mean number of particles at a state of given spin projection σ.

We must of course be aware of that the difference in momentum between two Bosons tend to be small and more pairs are observed than for Fermions.

Equation (28) also shows that the difference between Bosons and Fermions is similar in configuration space, the presence of one Boson at some point increases the probability that another Boson is close to that point.

Comparing the experimental distribution of two-particle space correlation calculated from eq. 28 or 30 with that from eq. 16, we could probe whether the first order phase transition has taken place. If no clasical density fluctuations and no quantum correlations, the particles will be emitted independently in space. Otherwise the correlation function becomes long range if particles are emitted at the critical point.

Figure 1 shows the dependence of the two particle space correlation on the distance between two particles (r) with and without a first order phase transition. Curve 1 represents the two-particle space correlation including the phase transistion ($a = 0$) and curve 2 without the phase transistion ($a/b \approx 4$). The temperature T=160 MeV is chosen. The calculations show that if parameter a is of the same magnitude as b, the difference of the two-particle emission pattern from that with the phase transition is clearly visible. Two-particle emission with small separation in distance dominates if there is no phase transition. The results are also calculated with a pure Boltzmann distribution with T =160 MeV and Bose distributions with two different temperatures. These results show no signal of a phase transition as expected. A possible single-particle momentum distribution (strong enhancement at low momentum and an enhancement at high momentum) which results in a long range correlation of two particles in space is also indicated in the figure. Experimentally, one

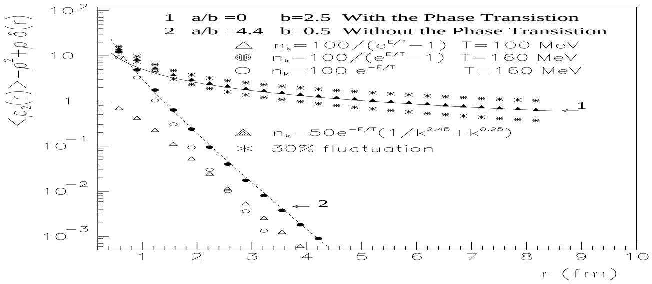

Figure 1: The dependence of two-particle space correlation function on the separation distance between two particles.

should search for the phase transition on an event-by-event basis, looking for significant fluctuations in a single-particle momentum distribution. Assuming that there is 30% fluctuation of the magnitude in the fitted single-particle momentum distribution at momentum less than 30 MeV/c due to detector resolution but keeping the function of the momentum distribution the same as that indicated by the solid triangle in the figure, the two-particle space correlation function will vary and the varied region is also indicated in the figure.

Figure 2 shows the single-particle momentum distributions which are used to calculate the two-particle space correlation. It demonstrates that what kind of shape of single-particle momentum distribution possibly indicate a phase transition.

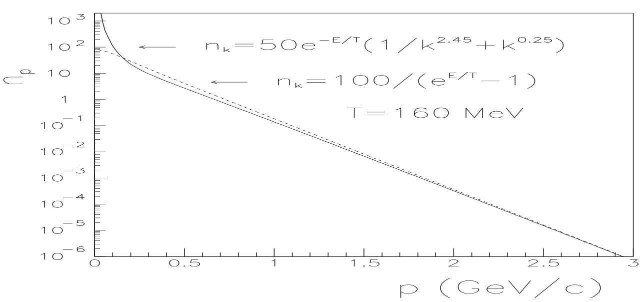

Figure 2: Comparison of two kind of single-particle momentum distributions used in the calculation of two-particle space correlation.

In eq.(26) the assumption of independent particles in different quantum states simplified the calculation of the two-particle space correlation. In fact this approximation can be removed and the two-particle space correlation can be deduced from single-particle

and two-particle momentum distributions.

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